3401 2011

1 (a) $y = e^x$

$$xy'' - (1+x)y' + y = 0 \qquad (\dagger)$$

Plug in: $xe^x - (1+x)e^x + e^x = 0$ ✓

$$xy'' - (1+x)y' + y = x^2 \qquad (*)$$

Let $y = ue^x$

$y' = u'e^x + ue^x$

$y'' = u''e^x + 2u'e^x + ue^x$ Plug into $(*)$.

$$x(u''e^x + 2u'e^x + ue^x) - (1+x)(u'e^x + ue^x) + ue^x = x^2$$

$$xu''e^x + 2u'e^x x + \cancel{xue^x} - u'e^x - \cancel{ue^x} - xu'e^x - \cancel{xue^x} + \cancel{ue^x} = x^2$$

$$xu'' + 2u'x - u' - xu' = x^2e^{-x}$$

$$xu'' + u'x - u' = x^2e^{-x}$$

$$xu'' + (x-1)u' = x^2e^{-x}$$

$z = u'$: $\quad xz' + (x-1)z = x^2e^{-x}$

$$z' + \frac{x-1}{x}z = xe^{-x}$$

$$\left[z e^{\int \frac{x-1}{x}dx} = z e^{[x - \ln x + C]} = \frac{ze^xC}{x}\right]$$

$$\Rightarrow \left[\frac{ze^xC}{x}\right]' = C$$

$$\frac{ze^xC}{x} = Cx + D$$

$\cancel{\Rightarrow z = Cxe^{-x} +}$

$\cancel{\Rightarrow u' = Cxe^{-x}}$

$\cancel{\Rightarrow u = -Cxe^{-x} + \int Ce^{-x}}$

$\cancel{= -Cxe^{-x} - Ce^{-x}}$

$$\frac{ze^x C}{x} = Cx + D$$

$$\Rightarrow z = x^2 e^{-x} + Exe^{-x}$$

$$\Rightarrow u' = x^2 e^{-x} + Exe^{-x}$$

$$\Rightarrow u = -x^2 e^{-x} + 2\int xe^{-x}\,dx - Exe^{-x} + \int Ee^{-x}\,dx$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx - Exe^{-x} - Ee^{-x} + F$$

$$=$$

$$ue^x = -x^2 - 2x - 2 - Ex - E + Fe^x$$

$$y = Fe^x + E(-1-x) - x^2 - 2x - 2$$

Let $G = -E$

$$= \underline{Fe^x + G(1+x) - x^2 - 2x - 2}.$$

(b) $y = \int_{C_i} e^{xt} f(t)\,dt$

$$x\frac{d^2y}{dx^2} - (1+x)\frac{dy}{dx} - y = 0$$

$$x\int t^2 e^{xt} f(t)\,dt - (1+x)\int t e^{xt} f(t)\,dt - \int e^{xt} f(t)\,dt = 0.$$

$$\int [xt^2 - (1+x)t - 1]\, e^{xt} f(t)\,dt = 0$$

want to say this is $\int \frac{d}{dt}[e^{xt} g]\,dt$

$$= \int (xe^{xt}g + e^{xt}g')\,dt$$

$\Rightarrow g = (t^2 - t)f$ $\quad\Rightarrow\quad \frac{g'}{g} = \frac{-t-1}{t^2-t}$

and $g' = (-t-1)f$

$$\frac{-t-1}{t^2-t} = \frac{-t-1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1} = \frac{(t-1)A+Bt}{t(t-1)} = \frac{(A+B)t-A}{t(t-1)}$$

$\Rightarrow A = 1$
$B = -2$

$$= \frac{1}{t} - \frac{2}{t-1}$$

$$\Rightarrow \ln g = \ln t - 2\ln(t-1) + C$$

$$\Rightarrow g = Ct(t-1)^{-2} = \frac{Ct}{(t-1)^2}$$

and $f = \dfrac{g}{t(t-1)} = \dfrac{Ct}{(t-1)^2 t(t-1)} = \dfrac{1 \cdot C}{(t-1)^3}$

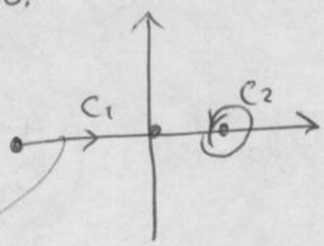
$$\Rightarrow y = C\int_{C_i} \frac{e^{xt}}{(t-1)^3}\,dt$$

Choose C_i s.t. $y \neq 0$
and $\left[e^{xt}\frac{t}{(t-1)^2}\right]_{t=-\infty}^{t=0} = 0$

$s=-t$ $\cancel{\int_{-1}^{\infty} \frac{e^{-sxs}}{(s+1)^3}\,ds}$

sings at $s=-1$

$$\cancel{= \oint_0^\infty \frac{e^{-xs}}{(s+1)^3}\,ds} + \cancel{\oint_{-1}^{0} \frac{e^{-xs}}{(s+1)^3}\,ds}$$



$$y = \int_{-\infty}^{0} \frac{e^{xt}}{(t-1)^3}\,dt = \int_0^\infty \frac{e^{-xt}}{(t+1)^3}\,dt$$

$$C_2 : \oint_{C_2} \frac{e^{xt}}{(t-1)^3}\,dt \propto \frac{d^2}{dt^2}e^{xt}\Big|_{t=1} = x^2e^x$$

$\because$ Cauchy.

2. (a) $\frac{dx}{dt} = 4x - x^2 - xy$ $\frac{dy}{dt} = -2y + xy$

$\frac{dy}{dx} = \frac{-2y + xy}{4x - x^2 - xy} = \frac{Q}{P}$ $\cancel{\frac{dy}{dx}}$

$Q = 0$: $y = 0$, $x = 2$

$P = 0$: $x(4 - x - y) = 0 \Rightarrow x = 0$, ~~or~~ $x + y = 4$
$y = 4 - x$

Critical pts where $P = Q = 0$, ie $(2,2)$
$(0,0)$
$(4,0)$

What are they like?

$$J = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} 4 - 2x - y & -x \\ y & -2 + x \end{pmatrix}$$

$J|_{(0,0)} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$. evals: $\begin{vmatrix} \lambda - 4 & 0 \\ 0 & \lambda + 2 \end{vmatrix}$ are $\underline{-2, 4}$.
real, different sign
$\underline{\text{saddle}}$

$J|_{(4,0)} = \begin{pmatrix} -4 & -4 \\ 0 & 2 \end{pmatrix}$ evals $-4, 2$. real, diff'ent sign
$\Rightarrow$ saddle.

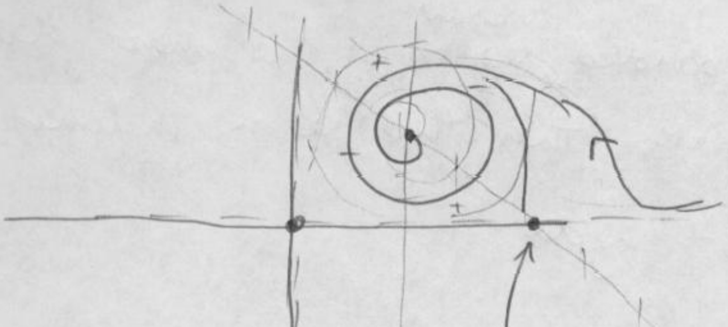
$J|_{(2,2)} = \begin{pmatrix} -2 & -2 \\ 2 & 0 \end{pmatrix}$ evals: $\begin{vmatrix} \lambda + 2 & 2 \\ -2 & \lambda \end{vmatrix}$ - $(\lambda)(\lambda + 2) + 4 = 0$

$\lambda^2 + 2\lambda + 4 = 0$

$\lambda = \frac{-2 \pm \sqrt{4 - 16}}{2}$

complex, real part neg.

$\Rightarrow$ stable spiral.



$y = mx$

$\frac{dy}{dx} = \frac{-2mx + xmx}{4x^2 - x^2 - xmx}$

careful about separatrices

$x = 4$

(b) $\dot{x} = x(x^2+y^2-2x-3) - y$

$\dot{y} = y(x^2+y^2-2x-3) + x$

- $r\dot{r} = x\dot{x} + y\dot{y} = \cancel{x^2(x^2+y^2-2x-3) - xy + y^2(x^2+y^2-2x-3) + y^2}$

$\cancel{r^2\dot{\theta} = x\dot{y} - y\dot{x}}$ $= r^2\cos^2\theta(r^2-2r\cos\theta-3) - r^2\cos\theta\sin\theta$

$+ r^2\cos\theta\sin\theta(r^2-2r\cos\theta-3) - r^2\sin^2\theta$

$= r^2[r^2\cos^2\theta - 2r\cos^3\theta - 3\cos^2\theta - \cos\theta\sin\theta$

$+ \cos\theta\sin\theta r^2 - 2r\cos^2\theta\sin\theta$

$-3\cos\theta\sin\theta - \sin^2\theta]$

- $r\dot{r} = x\dot{x} + y\dot{y} = x^2(x^2+y^2-2x-3) - xy + y^2(x^2+y^2-2x-3) + xy$

$r\dot{r} = r^2(r^2-2x-3)$

$\dot{r} = r(r^2-2r\cos\theta-3)$

$r^2\dot{\theta} = x\dot{y} - y\dot{x} = xy(x^2+y^2-2x-3) + x^2 - yx(x^2+y^2-2x-3) + y^2$

$= r^2$

$\rightarrow \dot{\theta} = 1$

PB thm: if $\exists$ a bounded invariant region in the phase plane with no eq^m points, then that region includes at least one limit cycle.

at $\underline{r=1}$, $\dot{r} = 1(1-2\cos\theta-3)$

$= -2-2\cos\theta \quad < 0$

$\underline{r=3}$ $\quad \dot{r} = 3(9-6\cos\theta-3) \quad > 0$.

and as $\dot{\theta} \neq 0$ in $1<r<3$, there are no critical points

$\Rightarrow$ <u>limit cycle</u>.

3. $$\frac{d^2x}{dt^2} + \varepsilon f(x)\frac{dx}{dt} + x = 0 \qquad (*)$$

(a) $x(t) = a\cos\theta + \varepsilon x_1(\theta) + \cdots$

$\theta = (1 + \varepsilon n_1 + \cdots)t$

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt}$$

$$= (1+\varepsilon n_1)\frac{dx}{d\theta}$$

$(*) \Rightarrow$ ~~$(1+\varepsilon n_1)^2 \frac{d^2x}{d\theta^2}$~~

$$(1+\varepsilon n_1)^2\left(-a\cos\theta + \varepsilon x_1''(\theta)\right) + \varepsilon f(x)(1+\varepsilon n_1)(-a\sin\theta + \varepsilon x_1'(\theta))$$

$$+ a\cos\theta + \varepsilon x_1(\theta) = 0.$$

$[\varepsilon^0]$ $-a\cos\theta + a\cos\theta = 0$ ✓

$[\varepsilon^1]$ $-2n_1 a\cos\theta + x_1''(\theta) - f(x)a\sin\theta + x_1(\theta) = 0.$

$\Rightarrow x_1'' + x_1 =$ ~~[illegible]~~

$2n_1 a\cos\theta + f(x)a\sin\theta$

~~[illegible]~~

We want the RHS to have no sin's or cos's $\because$ we don't want a periodic solution.

$\Rightarrow$ $\underline{n_1 = 0}$ gives no cos's.

$\times\sin$ $$\int_{-\pi}^{\pi} f(x)a\sin(\theta)^2 = 0.$$

$\frac{d^2x}{dt^2} + \varepsilon(x^2-\alpha$

If $f(x) = x^2 - \alpha^2$,

$$\int_{-\pi}^{\pi} \sin^2\theta\left[a^2\cos^2\theta - \alpha^2\right]d\theta = 0$$

$$\to a^2\int_{-\pi}^{\pi}\sin^2\theta\cos^2\theta\,d\theta - \alpha^2\int_{-\pi}^{\pi}\sin^2\theta\,d\theta = 0$$

$$a^2\frac{\pi}{4} - \alpha^2\pi = 0 \Rightarrow \underline{a = 2\alpha.}$$

(c) $y(t) = \frac{dx}{dt} + \varepsilon F(x)$ $\quad F(x) = \frac{1}{3}x^3 - \alpha^2 x$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} + \varepsilon\frac{dF}{dx}\frac{dx}{dt} \qquad \frac{dF}{dx} = \cancel{3}x^2 - \alpha^2$$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} + \varepsilon\frac{dx}{dt}(x^2-\alpha^2)$$

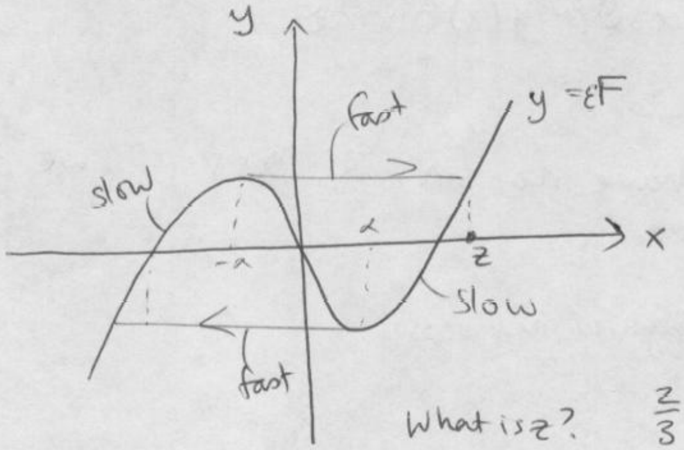
into (**):
$$\left.\begin{aligned}\frac{dy}{dt} &= -x\\ \frac{dx}{dt} &= y - \varepsilon F(x)\end{aligned}\right\}$$

$\varepsilon \gg 1 \Rightarrow \dot{x} \gg 1$ except where $y = \varepsilon F$

$F = \frac{1}{3}x^3 - \alpha^2 x$

$F' = \cancel{3}x^2 = \alpha^2$

$\to$ max/min at $x = \pm\alpha$



$F(\alpha) = \frac{1}{3}\alpha^3 - \alpha^3 = -\frac{2}{3}\alpha^3$

$F(-\alpha) = \frac{2}{3}\alpha^3$

What is z? $\frac{2}{3}\alpha^2 = \frac{1}{3}z^3 - \alpha^2 z$

$0 = \frac{1}{3}z^3 - \alpha^2 z - \frac{2}{3}\alpha^2$

$= z^3 - 3\alpha^2 z - 2\alpha^2$

we know

$= (z+\alpha)^2(z-2\alpha)$

$\to z = 2\alpha$

OK so now to work out period.

$$T = \int dt = \int \frac{dt}{dx}\,dx = \int \underbrace{\frac{dt}{dy}}_{\frac{1}{-x}}\;\underbrace{\frac{dy}{dX}}_{\substack{\text{small on fast parts} \\ = F' \text{ on slow part} \\ = x^2-\alpha^2}}\,dx$$

$2\alpha \to \alpha$
$= -2\alpha \to -\alpha$

$$\approx 2\int_{2\alpha}^{\alpha} -\frac{1}{x}(x^2-\alpha^2)\varepsilon \; dx$$

$$= (3 - 2\ln 2)\alpha^2 \varepsilon$$

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4. $$\frac{d^2x}{dt^2} + x = \varepsilon f\left(x, \frac{dx}{dt}\right) \qquad (*)$$

$T = \varepsilon t$

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + \frac{\partial}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}$$

$$\frac{d^2}{dt^2} \to \left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}\right)^2$$

Let $x = X_0 + \varepsilon X_1 + \cdots$

Into $(*)$: $$\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}\right)^2 (X_0 + \varepsilon X_1) + X_0 + \varepsilon X_1 = \varepsilon f\left(x, \frac{dx}{dt}\right).$$

$$\Rightarrow \frac{\partial^2 X_0}{\partial t^2} + \varepsilon 2\frac{\partial^2 X_0}{\partial t \partial T} + \varepsilon\frac{\partial X_1}{\partial t} + X_0 + \varepsilon X_1 = \varepsilon f$$

$[\varepsilon^0]$ $$\frac{\partial^2 X_0}{\partial t^2} + X_0 = 0 \quad \Rightarrow \quad X_0 = A(T)\sin(t + \phi(T))$$

$[\varepsilon^1]$ $$2\frac{\partial^2 X_0}{\partial t \partial T} + \frac{\partial X_1}{\partial t} + X_1 = f$$

$$\frac{\partial}{\partial T}\Big(A(T)\cos(t + \Phi(T))\Big) = A'\cos(t + \Phi(T)) - A\sin(t + \Phi(T))\,\Phi'$$

$$\Rightarrow \quad 2A'\cos(t+\Phi) - 2A\sin(t+\Phi)\,\Phi' + \frac{\partial x_1}{\partial t} + x_1 = f$$

Let $x = t + \Phi$:

$$2A'\cos x - 2A\sin x\,\Phi' + \frac{\partial x_1}{\partial t} + x_1 = f.$$

$\times \sin x$

$$0 = \int 2A\sin^2 x\,\Phi'\,dx + \int f\sin x\,dx - \int 2A'\cos x\sin x\,dx$$

$$= 2\pi A\Phi' + \int f\sin x\,dx \qquad (1)$$

$\times \cos x$

$$0 = \int f\cos x\,dx + \int 2A'\cos^2 x\,dx - \int 2A\sin x\cos x\,\Phi'\,dx$$

$$= \int f\cos x\,dx + A'\pi 2 \qquad (2)$$

$$f(x,\dot{x}) = x - \dot{x} \qquad\qquad x = A\sin(t+\Phi(T))$$

$$\dot{x} = A\cos(t+\Phi(T))$$

$$\rightarrow \quad f = A[\sin \cancel{t+} x \mp \cos x]$$

$$(1) \Rightarrow \frac{dA}{dT} = \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos x(\sin x - \cos x)\,dx$$

$$\frac{dA}{dT} = -\frac{A}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx = -\frac{A}{2}$$

$$\Rightarrow -\frac{1}{A}\,dA = \frac{1}{2}\,dT$$

$$\rightarrow \ln\frac{1}{A} = \frac{T}{2}$$

$$\frac{1}{A} = e^{T/2}$$

$$A = e^{-T/2} = e^{-\varepsilon t/2}$$

(2): $e^{-T/2}\dfrac{d\Phi}{dT} = -\dfrac{A}{2\pi}\int_{-\pi}^{\pi}\sin\chi(\sin\chi-\cos\chi)\,d\chi$

$$= -\frac{A}{2\pi}\int_{-\pi}^{\pi}\sin^2\chi$$

$$= -\frac{A}{2\pi}\pi = \frac{e^{-T/2}}{2}$$

$$\Rightarrow \frac{d\Phi}{dT} = \cancel{-\frac{A}{2}e^{-T/2}}\ \frac{1}{2}$$

$$\Rightarrow \Phi = \cancel{-\frac{A}{4}e^{T/2}+B}\ \frac{T}{2} + C$$

$$\cancel{\text{but } A = e^{-T/2}}$$

$$\Rightarrow \cancel{\sin(t+\Phi(T)) = \sin\left(t + -\frac{A}{\varepsilon t}e^{\varepsilon t/2} + B\right)}$$

$$\cancel{\Phi =} -$$

$$\Rightarrow \sin(t+\Phi) = \sin\left(t+\frac{\varepsilon t}{2}+c\right)$$

$$\Rightarrow x(t) \sim e^{-\varepsilon t/2}\sin\left(t+\frac{\varepsilon t}{2}+c\right)$$

$x(0)=1 \Rightarrow c = \frac{\pi}{2} \Rightarrow$ $\underline{x(t) \sim e^{-\varepsilon t/2}\cos\left(t+\frac{\varepsilon t}{2}\right)}$.

(c) $f(x,\dot x) = -\dot x(\dot x-\alpha)(\dot x-\beta) = -A\cos(t+\Phi)[A\cos(t+\Phi)-\alpha][A\cos(t+\Phi)-\beta]$

$$\frac{dA}{dT} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\chi[\quad\ldots$$

$$= -\frac{\alpha\beta}{2}A - \frac{3}{8}A^3$$

$$= \frac{A}{2}\left(|\alpha\beta| - \frac{3}{4}A^2\right) \quad \text{etc.}$$

stable if $\underline{\partial A/\partial T = 0}$.

5 (a) (i) If $I(x) = \int_0^T e^{-xt} f(t)\, dt$

and $f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^n \qquad t \to 0$

$$\Rightarrow \quad I(x) \sim \sum_{n=0}^{\infty} \frac{a_n (\alpha+n)!}{x^{n+\alpha+1}}$$

(ii) $K_0(x) = \int_1^{\infty} \frac{e^{-xt}}{\sqrt{t^2-1}}\, dt.$ Let $s = t-1$

$t = 1+s$

$$= \int_0^{\infty} \frac{e^{-xs} e^{-x}}{\sqrt{s^2+2s}}\, ds$$

$$= \frac{e^{-x}}{\cancel{\sqrt{2}}} \int_0^{\infty} \frac{e^{-xs}}{\sqrt{2s}\sqrt{1+s/2}}\, ds$$

$$\Rightarrow \quad f(t) = \frac{1}{\sqrt{2s}\sqrt{1+s/2}}\, ds$$

$$\sim \frac{1}{\sqrt{2}} s^{-1/2} \left(1 - \frac{s}{2\cdot 2} + \cdots\right) \quad \Rightarrow \quad \alpha = -\frac{1}{2}.$$

$a_0 = 1$

$a_1 = -\frac{1}{4}$

$$\Rightarrow \quad K_0(x) \sim \frac{e^{-x}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{a_n (-\frac{1}{2}+n)!}{x^{n+1/2}}$$

$$= \frac{e^{-x}}{\sqrt{2}} \left(\frac{(-\frac{1}{2})!}{x^{1/2}} - \frac{(\frac{1}{2})!\, \frac{1}{4}}{x^{3/2}} + \cdots \right)$$

$(-\frac{1}{2})! = \sqrt{\pi}$

$$= \frac{e^{-x}}{\sqrt{2}} \left(\frac{\sqrt{\pi}}{x^{1/2}} - \frac{1}{8} \frac{\sqrt{\pi}}{x^{3/2}} \right)$$

$$= e^{-x} \sqrt{\frac{\pi}{2x}} \left(1 - \frac{1}{8x}\right)$$

(iii). $\int_0^{\pi/4} e^{-x\sin^4 t}\sqrt{\sin t}\,dt \quad (*)$

Let $u = \sin^4 t$

$du = 4\sin^3 t\cos t\,dt$

$u = \sin^4\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)^4 = \frac{4}{16} = \frac{1}{4}$.

$\sin^{1/2} t = u^{1/8}$

$$\int_0^{1/4} e^{-xu}\frac{u^{1/8}}{4u^{3/4}(1-u^{1/2})^{1/2}}\,du$$

$\Rightarrow f(u) = \dfrac{u^{1/8}}{4u^{3/4}(1-u^{1/2})^{1/2}}$ $\qquad \frac{1}{8}-\frac{6}{8}$

$= \dfrac{u^{-5/8}}{4(1-u^{1/2})^{1/2}}$

as $u\to 0$, $u^{1/2}\to 0$

$= \dfrac{1}{u^{5/8}4}$ $\Rightarrow \alpha = -5/8$

$$\Rightarrow I(x) \sim \frac{1}{4}\frac{(-5/8)!}{x^{3/8}}$$

(b) $\int_0^{\pi}\cos(x\cos\theta)\sin^{2\nu}\theta\,d\theta$

$= \int_0^{\pi}\cos(x\cos\theta)\,e^{2\nu\log(\sin\theta)}\,d\theta$

$\Rightarrow \phi(\theta) = \ln\sin\theta$

$\phi'(\theta) = \cot\theta$

ϕ has a max at $\theta = \pi/2$.

Laplace's method gives where $\phi'' = -1$.

$$F \sim \sqrt{\frac{2\pi}{\nu|-1|}}\,\underbrace{\cos\left(x\cos\tfrac{\pi}{2}\right)}_{\cos 0 = 1}\,e^{\overbrace{2\nu\ln\sin\pi/2}^{=0}}$$

$= \sqrt{\dfrac{2\pi}{\nu}}$